USE OF SELF-SIMILAR NETS IN PROBLEMS WITH DISCONTINUITIES IN THEIR SOLUTIONS

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A preliminary transformation of coordinates to bring the equation into a form amenable to finite-difference methods is examined in relation to the heat conduction equation with a singularity in the boundary conditions.

To assess the accuracy of finite-difference approximations, the usual criterion is the smoothness of solutions of the equation [1]. In the case of discontinuities in the solution or its derivatives, it is necessary to go either to a very fine mesh, or to various methods of eliminating the singularities such as that, for example, of Volkov [2–4]. In some cases it is impossible to use a transformation of coordinates such that the solution becomes smooth enough in the new coordinates.

We will examine parabolic equations of the type

$$C\frac{\partial u}{\partial \tau}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial u}{\partial x}\right)+f,$$

$$0<x<X,\quad \tau>0, \tag{1}$$

for which the problem is posed with inconsistent initial and boundary conditions of the type

$$u|_{\tau=0}=0;$$
$$u|_{x=0}=\varphi(\tau);$$
$$u|_{x=X}=0. \tag{2}$$

The smoothness of the solution close to $x = 0$, $\tau = 0$ is disturbed depending on which of the quantities $\left.\frac{d^n\varphi}{d\tau^n}\right|_{\tau=+0}$ is the first to be nonzero.

We will first study the case when C and λ do not vanish at the point (0, 0), and are quite smooth. Then we may obtain in the first approximation for u the expression $\varphi(0)\,\mathrm{erfc}\left(\frac{x\sqrt{C}}{2\sqrt{\lambda\tau}}\right)$. The form of the first term for the asymptotic behavior of u at small τ and x suggests introduction of the new independent variable $\xi = x/(\tau)^{1/2}$; as a second approximation, to preserve parabolicity, we may put $\eta = \tau$. In the new independent variables, Eq. (1) takes the form

$$\eta C\frac{\partial u}{\partial \eta}=\frac{\partial}{\partial \xi}\left(\lambda\frac{\partial u}{\partial \xi}\right)+\frac{1}{2}C\xi\frac{\partial u}{\partial \xi}+\eta f, \tag{3}$$

where the dependence of λ and C on the independent variables is allowed for. In view of the fact that $\tau = \eta$, $x = \xi(\eta)^{1/2}$, a singular disturbance of the smoothness of the coefficients occurs only at small η (smoothness with respect to η is disturbed). Equation (3) has a singularity at $\eta = 0$, and the condition when $\eta = +0$ should not be imposed, since the behavior of the solution when $\eta = 0$ is determined completely by the quantity $\varphi(+0)$. The region of variation of the independent variables will now be $\eta > 0$, $0 < \xi < X/(\tau)^{1/2}$; because of the rapid fall of u with increase of ξ we may restrict examination at small η to the region $0 < \xi < \infty$. For Eq. (3), under the condition

$$u|_{\xi=0}=\varphi(\eta) \tag{4}$$

the solution close to the ξ axis will be no less smooth than φ; moreover, near this axis the solution will repeat the power features of φ at $\eta \sim 0$. At values of λ and C nonconstant with respect to x there is a disturbance of the smoothness of these quantities with respect to η, since smoothness only with respect to $(\eta)^{1/2}$ is preserved; the latter also holds for the solution.

In the construction of the finite-difference scheme for (3) a singularity arises near $\eta \sim 0$, since the equation is degenerate there. As has been pointed out above, at $\eta = +0$ one should solve the problem simply with respect to ξ, without taking account of η at values close to zero. The usual considerations that apply in constructing a scheme that approximates to the left and right sides of (3) with sufficient accuracy are not applicable, since near $\eta = 0$ the solution is smooth not with respect to ξ, η, but only with respect to ξ, $(\eta)^{1/2}$. It is undesirable to make the transformation $\eta \to (\eta)^{1/2}$, since it is convenient to preserve the uniformity of the net with regard to $\eta = \tau$, with the object of linking the solution later with description of the process outside the local disturbance of smoothness.

In accordance with the asymptotic form of the solution, in formulating the finite-difference scheme in the variables ξ, η, we should take account of smoothness with respect to ξ and of the near exponential behavior with respect to η. Because of the smoothness with respect to ξ and $(\eta)^{1/2}$, it is natural to take approximations to the differential expressions of the ordinary form with respect to ξ, and to carry out an approximation to differentiation with respect to η in such a way that it is exact for the asymptotic solution for $\eta \sim 0$, while going over to the usual relations at large η.

We will examine a six-point scheme approximating to (3). We will first consider the case of constant coefficients, but assume that to (3) we may add a heat source which, being smooth in the variables x, τ, will be smooth with respect to the variables ξ, $(\eta)^{1/2}$. In particular, we may thus take account of the deviation of λ and C from constancy, i.e., from their values at $\xi = 0$, $\eta = 0$, including nonlinear problems. We assume below that Eq. (1) is reduced by choice of scales to a form such that $\lambda(0,0) = C(0,0) = 1$.

Under the homogeneous conditions (4) there are solutions for (3) which behave as $\eta^{-k/2}$, $k = 1, 3, \ldots$, and correspond to an ordinary Boussinesq expansion, being unbounded when $\eta \to +0$. They characterize the fall in the influence of a local disturbance in the vicinity of $\eta = 0$, since along with such solutions is expressed the solution of the problem (1)–(4) for η exceeding the second coordinate of the source carriers. The general solution is obtained by superposition of such expansions. We will suppose that, because of smoothness of all the quantities, the approximation to the differential operator with respect to ξ on the right side of (3) does not produce difficulties, and leads to a sufficiently accurate approximation for each η, so that it may be considered that Eq. (3), approximated by the method of lines, has been solved. Further approximation reduces to sufficiently accurate replacement of this system of differential equations with respect to η by difference expressions with respect to η. The approximation relation for a first-order system with unknown vector functions is

$$\eta C \frac{\partial \bar{u}}{\partial \eta} = L_\xi \bar{u} + \bar{h}(\eta),$$
$$C\bar{u} = \{C_i u_i\}, \qquad (5)$$

where L_ξ is a difference approximation to the right side of (3), and $\bar{h} = \{h_i\}$, $\bar{u} = \{u_i\}$ are values of h and u at certain chosen nodes with respect to ξ. The dependence of the solution on η, as has been explained, is described for small η by semiintegral powers of η. The zero power corresponds to breakdown of the boundary value $u|_{x=0} \neq 0$. The influence of extraneous circumstances of the type of nonlinearity of the problem, the lack of constancy of λ and C is accounted for by the power +1/2, and the influence of attenuation of local disturbances will be described by negative powers, of which the first, i.e., −1/2, should be taken.

For (5) we construct a scheme of the type

$$a_j \bar{u}_j + b_j \bar{u}_{j-1} + c_j L_\xi \bar{u}_j + d_j L_\xi \bar{u}_{j-1} + e_j \bar{h}_j + f_j \bar{h}_{j-1} = 0, \quad (6)$$

where by $\bar{u}_j$, $\bar{h}_j$ we understand the value of the vectors $\bar{u}$ and $\bar{h}$ at the points $\eta = jl$ (l is the mesh size with respect to η). We assume that it must be accurate for solutions of the type indicated above when $\bar{h} = 0$. These solutions have the form $\eta^\alpha \bar{u}_\alpha$, where u_α satisfies the equation $L_\xi \bar{u}_\alpha = \alpha \bar{u}_\alpha$, it not being assumed that the homogeneous boundary condition is fulfilled when $\xi = 0$. Indeed, on the coefficients (6) we impose the conditions (C = const)

$$a_j + b_j = 0,$$
$$a_j \eta_j^{1/2} + b_j \eta_{j-1}^{1/2} + c_j \eta_j^{1/2} \cdot \frac{1}{2} + d_j \eta_{j-1}^{1/2} \cdot \frac{1}{2} = 0,$$
$$a_j \eta_j^{-1/2} + b_j \eta_{j-1}^{-1/2} - c_j \eta_j^{-1/2} \cdot \frac{1}{2} - d_j \eta_{j-1}^{-1/2} \cdot \frac{1}{2} = 0.$$

These equations determine a_j, b_j, c_j, d_j to an accuracy up to a general multiplier which plays no part when $\bar{h} = 0$. We may choose it in such a way that we have

$$a_j = -b_j = \frac{1}{2} \frac{j^{1/2} + (j-1)^{1/2}}{j^{1/2} - (j-1)^{1/2}},$$
$$c_j = d_j = \frac{1}{2}.$$

When $\eta \to \infty$ this scheme goes over to the symmetrical implicit scheme of [1]. We have the two coefficients e_j, f_j to take into account the influence of $\bar{h}$, and they must be determined by assigning the form of $\bar{h}$, as determined from the conditions of the original problem. Quite frequently, in particular, $\bar{h}$ will have the form $\eta^\alpha u_\beta$.

In spite of our having reduced the problem of determining the singularities of the solution to the usual finite-difference form, the boundary with respect to ξ where the equations are being examined is very large $(X/(\tau)^{1/2})$. In view, however, of the rapid decrease in a disturbance due to a singularity at the boundary, the behavior will only be appreciable in a certain region, depending on the required accuracy of the calculations. We may determine the boundary of this region on the basis of the upper bound solution $u_0 = M\,\mathrm{erfc}\,(\xi/2)$, i.e., we take the region to be $0 < \xi < \xi_0$, where ξ_0 is a root of the equation $u_0(\xi) = \varepsilon$, and ε is the required accuracy. For small ε the use of the asymptote gives the estimate $\xi_0 \simeq \varphi(\varepsilon) \ln^{1/2}(1/\varepsilon)$ for ξ_0, where $\varphi(\varepsilon)$ is bounded above and below by positive constants. At the boundary $\xi = \xi_0$ we may impose a condition of the first kind. If we change over to conditions of the third kind, $\left(u + \alpha \frac{\partial u}{\partial \xi}\right)\Big|_{\xi=\xi_0} = 0$, then by requiring that the standard solution, which is the main term of the asymptote for the exact solution, e.g., erfc $(\xi/2)$, satisfies this condition, we find α. A solution different from the standard will not satisfy this boundary condition, and to estimate the error in the solution we may use the value of the discrepancy when the minor term of the asymptote of the solution is put into the boundary condition. Direct calculation shows that in an exact treatment we may underestimate somewhat the value of ξ_0 with the given accuracy.

We will examine the question of joining the solution near a singularity with the solution outside the region Ω $(\eta \geq 0,\ 0 \leq \xi \leq \xi_0)$. Let the solution outside this region be found by the use of standard six-point schemes, taking account only of equations for which all six points lie outside the self-similar region. At the boundaries of the regions, in an exact differential problem, we must assign conditions of the fourth kind. We will describe how these conditions may be taken into account in composing the finite-difference equations.

Let there be a nonself-similar net with mesh sizes h_1 and $l_1 = l$, $x_l = ih_1$, $\tau_j = l_j$ outside region Ω; if all the six points entering into a certain finite-difference equation are not contained in Ω, this equation is taken into account. If of the six points (ij), (i ± 1,j), (i,j − 1),

$(i \pm 1, j-1)$ only the point $(i-1, j)$ of the first three falls inside Ω, we proceed as follows. An equation is taken for these six points, the point $(i-1, j)$ being considered fictitious, i.e., the value of u at that point does not enter into the final description of the field of u. According to the points (i, j), $(i \pm 1, j)$ we construct an interpolation polynomial on the line $\tau = jl$. Then an equation is taken in Ω which contains the three points closest to ξ_0 with respect to ξ, points with $\eta = jl$, $\eta = (j-1)l$. Then equating values of u at the two points closest to ξ_0 with $\eta = \tau = jl$ in Ω to their values obtained from the interpolation polynomial in the nonself-similar net, we may exclude from the four relations obtained the value of u at the point $(i-1, j)$, as well as the values at the two points of the net nearest to ξ_0. Then of the six unknown quantities with $\tau = \eta = jl$ there remain only three, i.e., in the nonself-similar net, the values of u at the points (i, j) and $(i+1, j)$, and in the self-similar one—the value of u at the third point of the self-similar net from the boundary $\xi = \xi_0$. These three values are connected by one relation, into which enter also the values when $\tau = \eta = (j-1)l$, possibly at fictitious points. We will consider the values of the quantities in both nets at $(j-1)$ to be already known, in accordance with the parabolic nature of the problem, and in the event of it being impossible to use the quantities u at the fictitious points, these values are recovered from the compatibility conditions.

If more than one point in the nonself-similar net with $\tau = jl$ falls in Ω for certain equations, then the pertinent operation is not carried out, and the equations are not taken into account. Of the equations in Ω we take account only of those which do not contain values of u at the excluded points. Thus, for each j we have, for determination of u, a system of equations with a tridiagonal matrix, whose solution does not present special difficulties.

We note that analogous operations may also be performed to eliminate singularities of any specific form, if we introduce a new system of coordinates such that the main part of the singularity will be a smooth function of these coordinates.

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